A topological limit of gravity admitting an $SU(2)$ connection formulation

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We study the Hamiltonian formulation of the generally covariant theory defined by the Lagrangian 4-form $\mathscr{L} = e_I \wedge e_J \wedge F^{IJ}(\omega)$ where e^I is a tetrad field and F^{IJ} is the curvature of a Lorentz connection ω^{IJ}. This theory can be thought of as the limit of the Holst action for gravity for the Newton constant $G \to \infty$ and Immirzi parameter $\gamma \to 0$, while keeping the product $G\gamma$ fixed. This theory has for a long time been conjectured to be topological. We prove this statement both in the covariant phase space formulation as well as in the standard Dirac formulation. In the time gauge, the unconstrained phase space of theory admits an $SU(2)$ connection formulation which makes it isomorphic to the unconstrained phase space of gravity in terms of Ashtekar-Barbero variables. Among possible physical applications, we argue that the quantization of this topological theory might shed new light on the nature of the degrees of freedom that are responsible for black entropy in loop quantum gravity.

I. INTRODUCTION

The remarkable fact that general relativity can be described in terms of fields of the kind used in Yang-Mills theories [1] renewed hope on the possibility of defining a background independent approach to the canonical quantization of gravity. It was later realized [2] that a simple canonical transformation could be used to replace the (complex) self-dual variables (or Ashtekar variables) by real $SU(2)$ variables (the so-called Ashtekar-Barbero variables) more suitable for the definition of the quantization program. Holst's action was first introduced in [3] as a covariant formulation of gravity directly leading to the real $SU(2)$ connection formulation upon canonical analysis. The action takes the following form

$$I_H = \frac{1}{8\pi G}\int {}^*(e^I \wedge e^J) \wedge F_{IJ}(\omega) + \frac{1}{8\pi G\gamma}\int e^I \wedge e^J \wedge F_{IJ}(\omega), \tag{1}$$

where e^I is the tetrad 1-forms describing the gravitational field, F^{IJ} are the curvature 2-forms of a Lorentz connection ω_μ^{IJ}, G is Newton's constant, and γ is the co-called Immirzi parameter [4]. The $*$ denotes the duality operator acting on the internal indices $IJKL$. The first term is the standard Palatini action of general relativity, while second term can be shown not to affect the classical equations of motion. The reason for this is that $\delta_\omega I_H = 0$ is independent of γ, and implies the connection to be the uniquely defined torsion free connection compatible with e: $\omega = \omega(e)$. The second term contribution to the equation $\delta_e I_H = 0$ vanishes identically when evaluated on $\omega(e)$ due to the Riemann tensor identity $R_{[\mu\nu\rho]\sigma} = 0$. The canonical formulation of the Holst action leads in fact to a family of $SU(2)$ connection formulations of the phase space of general relativity labelled by γ: all of them related by canonical transformations.

However, in the quantum theory [5] the canonical transformations relating different connection formulations appear not to be unitarily implemented. For instance the spectra of geometric operators depend on the combination $G\gamma$. Formally speaking, the off shell contributions of the second term in the Holsts action have a non trivial effect on amplitudes in the path integral formulation of quantum gravity. This has an important effect in the computation of black hole entropy in

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LQG [6]. There is complete agreement on the universal dependence of entropy on fundamental couplings; more precisely, the leading order in the entropy formula is given by

$$S_{BH} = \frac{\gamma_0 a_{BH}}{4G\gamma\hbar}, \tag{2}$$

where a_{BH} is the macroscopic black hole area and γ_0 is a dimensionless constant.

This motivates to consider the limit $G \to \infty$ and $\gamma \to 0$ while keeping the product $G\gamma = G_0\gamma_0 = constant$. In such a limit we have

$$I_H \to I_0 = \frac{1}{G_0\gamma_0}\int e^I \wedge e^J \wedge F_{IJ}(\omega), \tag{3}$$

where I_0 is a theory thought to be topological and hence to lack of local physical degrees of freedom. In this work we will study in detail the classical properties of I_0 by performing its canonical analysis (the treatment of [8] excluded this singular case). We shall show that the previous limit is indeed a singular limit where (in the absence of boundaries) physical degrees of freedom are lost in the limiting procedure. In the absence of boundaries I_0 is a topological theory. However, we will show that non trivial degrees of freedom can arise in the presence of space time boundaries. Therefore, this singular limit should be relevant at least for a different understanding of nature of black hole entropy in LQG. This is expected to be so from the fact that the black hole entropy depends on the special combination of couplings $G\gamma = G_0\gamma_0$, and from the fact that all the degrees of freedom counted in the calculation of black hole entropy in LQG are boundary degrees of freedom living on the black hole horizon.

II. THE MODEL

From now on we concentrate on the study of the model defined by the action I_0 which, taking $G_0\gamma_0 = 1$ and putting all the indices, takes the form (this action has been already considered in [17])

$$I_0 = \int e^I \wedge e^J \wedge F_{IJ}(\omega). \tag{4}$$

The equations of motion of the previous theory are quite simple: variations $\delta_\omega I_0 = 0$ yield

$$d^\omega_{[\mu}(e^I_\nu e^J_{\rho]}) = 0 \;\Leftrightarrow\; d^\omega(e^I \wedge e^J) = 0 \;\Leftrightarrow\; d^\omega e^I = 0, \tag{5}$$

identical to the connection variations of the Hilbert-Palatini action. Variations of I_0 with respect to the tetrad $\delta_e I_0 = 0$ yield

$$\epsilon^{\alpha\beta\gamma\delta} e_{\beta I} F^{IJ}_{\gamma\delta} = 0 \;\Leftrightarrow\; e^J \wedge F_{IJ} = 0. \tag{6}$$

These last field equations are trivially satisfied once (5) hold as a consequence of the sixteen Riemann tensor identities $R_{[\mu\nu\lambda]\rho} = 0$. This seems to imply that our theory admits a much larger set of classical solutions that I_H. However, this naive conclusion is indeed false. The reason is that the action I_0 has also a larger group of local (gauge) symmetries. This can be made clear by a systematic study of the phase space of the model. In the rest of this section we perform the canonical analysis of this action. In the next subsection we study its phase space structure from the covariant phase space perspective [10]. In Subsection II B we perform Dirac canonical analysis in a manifestly Lorentz invariant manner. Finally, in Subsection II C we study the Dirac formulation in the time-gauge which allow us to introduce a phase space parametrization in terms of $SU(2)$ connection variables.

A. Analysis in the covariant phase space

Before carrying out formally the canonical analysis, it is worthwhile to perform a covariant phase space analysis. By doing this we will obtain the symplectic potential and symplectic 2-form of our field theory Eq.(4), also we expect acquiring some rapid qualitative properties which will provide

guidelines for the Dirac canonical analysis that follows. The following analysis adopts the notations and conventions in [10] and the general theory is found in the references therein.

Let $(\bar{\delta}e^I, \bar{\delta}\omega^{IJ})$ be any variation of the configuration variables, then the corresponding variation of the action is

$$\bar{\delta}I_0 = \int \left[2e^I \wedge F_{IJ} \wedge \bar{\delta}e^J + e^I \wedge e^J \wedge d^\omega \bar{\delta}\omega_{IJ}\right] \tag{7}$$

$$= \int \left[2(e^I \wedge F_{IJ}) \wedge \bar{\delta}e^J - d^\omega(e^I \wedge e^J) \wedge \bar{\delta}\omega_{IJ}\right] + \int d(e^I \wedge e^J \wedge \bar{\delta}\omega_{IJ}), \tag{8}$$

where in the second line, the two terms in the first integral yield the equations of motion Eqs.(5) and (6), while the second integral gives the symplectic potential

$$\Theta(\bar{\delta}) = \int_\Sigma e^I \wedge e^J \wedge \bar{\delta}\omega_{IJ}. \tag{9}$$

The integration above is carried out on any time like surface Σ, and the pull back on Σ of $e^I \wedge e^J \wedge \bar{\delta}\omega_{IJ}$ is understood. Now let Γ_{cov} be the covariant phase space consisting of all the solutions of equations of motion (5) and (6). Let also δ be any tangent vector to Γ_{cov}, that is, let $(\delta e^I, \delta\omega^{IJ})$ be any displacement between two neighboring solutions in Γ_{cov}. We can calculate the pull-back of symplectic potential on it, $\Theta(\delta) = \int_\Sigma e^I \wedge e^J \wedge \delta\omega_{IJ}$. In fact, variating the equations of motion (5) in Γ_{cov}, we have

$$\begin{aligned} &d(\delta e^I) + \delta\omega^I{}_J \wedge e^J + \omega^I{}_J \wedge \delta e^J = 0 \\ &\Rightarrow e^J \wedge \delta\omega_{IJ} = d(\delta e_I) + \omega_{IJ} \wedge \delta e^J. \end{aligned} \tag{10}$$

Substituting this into the pulled-back symplectic potential $\Theta(\delta)$, we have

$$\begin{aligned} \Theta(\delta) &= \int_\Sigma e^I \wedge e^J \wedge (\delta\omega_{IJ}) \\ &= \int_\Sigma \left[e_I \wedge d(\delta e^I) + e^I \wedge \omega_{IJ} \wedge \delta e^J\right] = \int_\Sigma \left[e_I \wedge d(\delta e^I) + \omega_{JI} \wedge e^I \wedge \delta e^J\right] \\ &= \int_\Sigma \left[e_I \wedge d(\delta e^I) - de_I \wedge \delta e^I\right] = -\int_\Sigma d(e_I \wedge \delta e^I) = -\int_{\partial\Sigma} e_I \wedge \delta e^I, \end{aligned} \tag{11}$$

where the equations of motion (5) is used in the beginning of the third line, and in the final step $e_I \wedge \delta e^I$ is in fact pulled back on $\partial\Sigma$. Thus the symplectic potential, pulled back on Γ_{cov}, turns out to be a total derivative and hence is a boundary term. The symplectic form Ω, defined as the pull back to Γ_{cov} of the curl of the symplectic potential, is therefore

$$\Omega(\delta_1, \delta_2) = -2\int_{\partial\Sigma} \delta_{[1}e_I \wedge \delta_{2]}e^I, \tag{12}$$

where the infinitesimal displacements δ_1 and δ_2 in Γ_{cov} are considered also as the tangent vectors of Γ_{cov}. We see that in the case $\partial\Sigma = \emptyset$ (or more generally for restrictive boundary conditions fixing $\delta e = 0$) the presymplectic form (12) is identically zero. This implies that all variations δ_1 or δ_2 are degenerate directions of the presymplectic form and hence should be regarded as pure gauge. Thus (locally) all solutions in Γ_{cov} are physically equivalent and we have no local degrees of freedom. This analysis shows that (4) is a topological field theory.

In the cases where Σ has boundaries (and depending on the boundary conditions) the symplectic form can be non-zero. For example at the presence of a black hole. In such cases, Eq.(12) has non-trivial contribution on the horizon.

B. Canonical analysis without time gauge

In this section we perform the canonical analysis following Dirac's method [11]. From now on we assume the spacetime manifold to be of topology $M = \Sigma \times \mathbb{R}$ where Σ is a compact three manifold. We choose coordinates (t, x^a) such that the surfaces Σ_t defined by $t = \textit{constant}$ defines a foliation of M, and x^a with $a = 1, 2, 3$ are local coordinates on Σ_t from now on denoted simply by Σ. The results presented here were partially investigated in [12]. The complete analysis including much more details than in this paper can be found in [13].

1. Primary and secondary constraints

Applying the 3+1 decomposition $e_t^I = Nn^I + N^a e_a^I$ to the action gives

$$
\begin{aligned}
I_0 &= -\frac{1}{2}\int \epsilon^{abc}\epsilon_{IJKL}(e_t^I e_a^J {}^*F_{bc}^{KL} + e_a^I e_b^J {}^*F_{tc}^{KL}) \\
&= \int \tilde{N}\Pi_{IK}^b \Pi^{cK}{}_J {}^*F_{bc}^{IJ} - N^b \Pi_{IJ}^a F_{ab}^{IJ} + \omega_t^{IJ} D_a(\Pi_{IJ}^a) - \dot{\omega}_a^{IJ}\Pi_{IJ}^a,
\end{aligned}
\tag{13}
$$

where $\tilde{N} := -N^2/e$, $e = \det(e_{\mu I})$ and $\Pi_{IJ}^a = \epsilon^{abc} e_{bI} e_{cJ}$. By performing the Legendre transformation, one obtains the Hamiltonian

$$
H = \int \tilde{N}\Pi_{IK}^a \Pi^{bK}{}_J {}^*F_{ab}^{IJ} + N^a \Pi_{IJ}^b F_{ab}^{IJ} - \omega_t^{IJ} D_a \Pi_{IJ}^a + \lambda_a^I M_I^a + \lambda_a^{IJ}(\Pi_{IJ}^a - \epsilon^{abc} e_{bI} e_{cJ}), \tag{14}
$$

where $\tilde{N}, N^a, \omega_t^{IJ}, \lambda_a^I$, and λ_a^{IJ} are Lagrange multipliers imposing the primary constraints

$$M_I^a \approx 0, \tag{15}$$
$$C_{IJ}^a := \Pi_{IJ}^a - \epsilon^{abc} e_{bI} e_{cJ} \approx 0, \tag{16}$$
$$\mathcal{S} := \Pi_{IK}^a \Pi^{bK}{}_J {}^*F_{ab}^{IJ} \approx 0 \quad \text{(scalar constraint)}, \tag{17}$$
$$\mathcal{V}_a := \Pi_{IJ}^b F_{ab}^{IJ} \approx 0 \quad \text{(vector constraint)}, \tag{18}$$
$$\mathcal{G}_{IJ} := D_a \Pi_{IJ}^a \approx 0 \quad \text{(Lorentz-Gauss law)}, \tag{19}$$

Here our phase space is parametrized by the canonical pairs (M_I^a, e_a^I), and $(\Pi_{IJ}^a, \omega_a^{IJ})$.

Now we start studying the consistency conditions of the primary constraints. The consistency conditions for Eqs.(15) $M_I^a \approx 0$ imply

$$\lambda_b^{JK} \epsilon^{abc} e_{cJ} \approx 0, \tag{20}$$

which can be shown to fix 12 out of the 18 Lagrange multipliers λ_a^{IJ}. This suggests that one can re-combine the 18 constraints C_{IJ}^a into two groups: one consists of 6 constraints that commute with M_I^a, leading to give 6 secondary constraints, and the other consists of 12 constraints that do not commute with M_I^a, fixing the 12 multipliers λ_a^I. The first group is given precisely by the (often called) simplicity constraints

$$\Phi^{ab} := \frac{1}{2}\epsilon^{IJKL}\Pi_{IJ}^a \Pi_{KL}^b = \mathrm{Tr}\left({}^*\Pi^a \Pi^b\right) \approx 0, \tag{21}$$

The second group is denoted by $\Xi^l \approx 0$, l running from 1 to 12. We will calculate the consistency conditions of Φ^{ab} and Ξ^l instead of those of C_{IJ}^a.

To evolve Φ^{ab} in time, we notice two things that can simplify the calculation. First, the Gauss-Lorentz law constraints $\mathcal{G}_{IJ} \approx 0$ are generators of Lorentz transformation on the internal indices, so that they commute with any constraints carrying no internal indices, such as Φ^{ab}. Second, the vector constraints $\mathcal{V}_a \approx 0$ and the Gauss law can be combined to give generators of spatial diffeomorphism $\tilde{\mathcal{V}}_a := \mathcal{V}_a - \omega_a^{IJ}\mathcal{G}_{IJ}$, who commute weakly with Φ^{ab}. The consistency conditions of $\Phi^{ab} \approx 0$ can be written in terms of smeared quantities

$$
\begin{aligned}
\dot{\Phi}^{ab}[\lambda_{ab}] &\approx \{\Phi^{ab}[\lambda_{ab}], \mathcal{S}[\tilde{N}]\} = \int\int \epsilon^{IJKL}\lambda_{ab}\Pi_{KL}^b \{\Pi_{IJ}^a, F_{cd}^{MN}\}\tilde{N}\left({}^*\Pi^c\Pi^d\right)_{MN} \\
&= 4\int \lambda_{ab}{}^*\Pi^{bIJ} D_c\left(\tilde{N}\left({}^*\Pi^c\Pi^a\right)_{IJ}\right) \approx -4\int \tilde{N}\lambda_{ab}\mathrm{Tr}\left({}^*\Pi^a {}^*\Pi^c D_c \Pi^b\right) \\
&= 4\int \tilde{N}\lambda_{ab}\mathrm{Tr}\left(\Pi^a\Pi^c D_c\Pi^b\right) := \chi^{ab}[\tilde{N}\lambda_{ab}],
\end{aligned}
\tag{22}
$$

where $\Phi^{ab}[\lambda_{ab}] = \int \lambda_{ab}\Phi^{ab}$ and similarly for $\mathcal{S}[\tilde{N}]$ and $\chi^{ab}[\tilde{N}\lambda_{ab}]$. Here we used the Gauss law constraint and the fact that $\mathrm{Tr}(\Pi^{(a}\Pi^{|c|}\Pi^{b)}) \approx 0$ by virtue of Eq.(16). This leads to 6 secondary constraints

$$\chi^{ab} := \mathrm{Tr}\left(\Pi^{(a}\Pi^{|c|}D_c\Pi^{b)}\right) = -\mathrm{Tr}\left({}^*\Pi^{(a\,*}\Pi^{|c|}D_c\Pi^{b)}\right) \approx 0. \tag{23}$$

We do not bother to care about the exact expression of Ξ^l. The consistency conditions $\dot{\Xi}^l$ fix the multipliers of constraints Eq.(15), the 12 λ^I_a and no secondary constraint arises. As for the 12 multipliers of Ξ^l, they are in fact just those that are fixed in Eq.(20). Thus $\Xi^l \approx 0$ fall in the second class together with $M^a_I \approx 0$, and they discard the 12 degrees of freedom carried by e^I_a. The evolution of χ^{ab} does not lead further constraints.

2. *Reducibility of the constraints*

At this stage, a naive counting would yield a negative number of degrees of freedom. This is a clear indication that not all constraints are independent: there is reducibility in the constraint system. In fact we will now prove that the scalar and vector constraints are in fact implied by the Gauss-Lorentz law and the secondary constraints Eq.(23). To see this let us express the relevant constraints ($\mathcal{G}_{IJ}$, χ^{ab}, $\mathcal{S}$ and $\mathcal{V}_a$) in terms of the tetrad components, with the help of Eq.(16). In particular, on one hand for $\mathcal{G}_{IJ}$ and χ^{ab},

$$\mathcal{G}_{IJ} \approx D_a\left(\epsilon^{abc} e_{bI} e_{cJ}\right) = \epsilon^{abc} e_{b[I} D_a e_{cJ]}, \tag{24}$$

$$\begin{aligned}
\chi^{ab} &= {}^*\Pi^{(aK}_I {}^*\Pi^{|c|}_{KJ} D_c \Pi^{bJI} \approx e^2 \left(e^t_I e^{(aK} - e^{tK} e^{(a}_I\right)\left(e^t_K e^{|c|}_J - e^t_J e^{|c|}_K\right) D_c\left(\epsilon^{b)fg} e^J_f e^I_g\right) \\
&= e^2\left(e^t_I e^c_J g^{t(a} - e^t_I e^t_J g^{c(a} - g^{tt} e^c_J e^{(a}_I + g^{tc} e^t_J e^{(a}_I\right) D_c\left(\epsilon^{b)fg} e^J_f e^I_g\right) \\
&= e^2\left(e^t_I g^{t(a} - g^{tt} e^{(a}_I\right)\epsilon^{b)cd} D_c e^I_d = e^2 N^{-2}\left(e^t_I N^{(a} + e^{(a}_I\right)\epsilon^{b)cd} D_c e^I_d\,.
\end{aligned} \tag{25}$$

Here $g^{\mu\nu} := e^{\mu I} e^\nu_I$ is the inverse spacetime metric, and it is related with the lapse and the shift by $g^{at} = N^a/N^2$ and $g^{tt} = -1/N^2$ (see §2.3 of [15]). One can show[1] that the previous twelve constraints are equivalent to

$$\mathcal{C}^{aI} := \epsilon^{abc} D_b e^I_c \approx 0. \tag{26}$$

Applying D_a to these constraints, we obtain

$$D_a \mathcal{C}^{aI} = \epsilon^{abc} D_a D_b e^I_c = \epsilon^{abc} e_{cJ} F^{IJ}_{ab} \approx 0. \tag{27}$$

On the other hand, the constraints $\mathcal{S}$ and $\mathcal{V}_a$, can be written as

$$\mathcal{S} = {}^*\Pi^a_{IK} \Pi^{bK}{}_J F^{IJ}_{ab} \approx e\left(e^t_I e^a_K - e^t_K e^a_I\right)\epsilon^{bcd} e^K_c e_{dJ} F^{IJ}_{ab} = -e\epsilon^{abc} e^t_I e_{cJ} F^{IJ}_{ab}\,, \tag{28}$$

$$\mathcal{V}_a = \Pi^b_{IJ} F^{IJ}_{ab} \approx \epsilon^{bcd} e_{cI} e_{dJ} F^{IJ}_{ab} = \frac{1}{2}\epsilon^{bcd} e_{cI} e_{dJ} \epsilon_{abf} \epsilon^{fgh} F^{IJ}_{gh} = -e_{aI} \epsilon^{bcd} e_{bJ} F^{IJ}_{cd}\,, \tag{29}$$

both of which vanish as a consequence of (27). Therefore, the constraints $\mathcal{S}$ and $\mathcal{V}_a$ are implied by the constraints (16), (19), and (23). Thus they can be safely removed from the Hamiltonian (14). This operation preserves the constraint surface as well as the trajectories of motion, at the harmless cost of certain modifications of multipliers of $\mathcal{G}_{IJ}$ and χ^{ab}. However certainly one has to add χ^{ab} to the Hamiltonian, which now reads

$$H = \int \kappa^{IJ} \mathcal{G}_{IJ} + \kappa_{ab} \chi^{ab} + \gamma_{ab} \Phi^{ab} + \gamma_l \Xi^l + \lambda^I_a M^a_I. \tag{30}$$

Now we are ready to classify the constraints. Our analysis so far shows that $\mathcal{G}_{IJ}, \Phi^{ab}$ and χ^{ab} are first class; while M^a_I and Ξ^l are second class. Thus for the 60 dimensional unconstrained phase space parametrized by (M^a_I, e^I_a), and $(\Pi^a_{IJ}, \omega^{IJ}_a)$ we have 18 first class constraints and 24 second class constraints which yields zero local degrees of freedom as expected from the analysis of Subsection II A.

Further insight into the nature of this topological model will be gained by repeating this analysis using the partial gauge fixing of the Lorentz symmetry known as the time gauge. This will reduce the internal gauge group from $SO(3,1)$ to $SO(3)$, and will make the relationship with gravity more explicit.

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[1] A key step in showing that the transformation matrix from (24) and (25) is non degenerate is to write down the inverse tetrad component in terms of the tetrad component: $e^a_I = \frac{1}{2e}\epsilon^{abc}\epsilon_{JIKL} e^J_t e^K_b e^L_c$.

C. Canonical analysis under time gauge

1. The Hamiltonian and the primary constraints under time gauge

To redo the analysis under time gauge, let us return to the Hamiltonian (14). The time gauge condition is defined by identifying the zero-th component of the tetrad $e_{\mu 0}$, with $n_\mu = (-N, 0, 0, 0)$, the the co-normal of the space-like hyper-surfaces of $3+1$ foliation of spacetime[2]. This is equivalent of imposing $n_I e^I_a = 0$, where $n_I = e_{\mu I} n^\mu$. One can also prove that under this condition, $n_I = (1, 0, 0, 0)$. Therefore, we can impose the time gauge condition by adding to the list of primary constraints Eqs.(15)—(19)

$$e^0_a \approx 0. \tag{31}$$

They give 6 second class constraints together with $M^a_0 \approx 0$ which can be solved directly in order to get rid of e^0_a and $M^a_0 \approx 0$ from the analysis. In this process the phase space is reduced to the canonical pairs (M^a_i, e^i_a) and $(\Pi^a_{IJ}, \omega^{IJ}_a)$, and the action (13) becomes:

$$I_0 = \int -\frac{N}{2} \frac{\epsilon_i{}^{jk} E^a_j E^b_k F^{i0}_{ab}}{\sqrt{\det E}} - N^b E^a_i F^i_{ab} + \omega^{i0}_t D_a(\Pi^a_{i0}) + \omega^{ij}_t D_a(\Pi^a_{ij}) - \dot{A}^i_a E^a_i, \tag{32}$$

where we used the definitions $E^a_i := \frac{1}{2}\epsilon_i{}^{jk}\Pi^a_{jk}$, and $A^i_a := -\frac{1}{2}\epsilon^i{}_{jk}\omega^{jk}_a$. If in addition we define $K^i_a := \omega^{0i}_a$ the previous expression becomes

$$I_0 = \int -E^a_i \dot{A}^i_a + \Pi^a_i \dot{K}^i_a - H, \tag{33}$$

where the Hamiltonian takes the (perhaps) more familiar form

$$H = \int \frac{N}{2} \frac{\epsilon_i{}^{jk} E^a_j E^b_k \mathscr{D}_a K^i_b}{\sqrt{\det E}} + N^b E^a_i \mathscr{F}^i_{ab} + N^i \epsilon_{ijk} E^{aj} K^k_a + M^i \mathscr{D}_a E^a_i + \lambda^i_a C^a_i + \rho^i_a M^a_i + \gamma^i_a \Pi^a_{i0}, \tag{34}$$

where $\mathscr{D}$ and $\mathscr{F}$ are the covariant derivative and curvature of the $SU(2)$ connection A^i_a, and $N, N^a, N^i, M^i, \lambda^a_i, \rho^a_i$, and γ^i_a are Lagrange multipliers. The Poisson brackets among the basic variables are

$$\{E^a_i(x), A^j_b(y)\} = \delta^a_b \delta^j_i \delta^3(x, y), \quad \{K^i_a(x), \Pi^b_{0j}(y)\} = \delta^b_a \delta^i_j \delta^3(x, y), \tag{35}$$

and the primary constraints are

$$M^a_i \approx 0 \tag{36}$$

$$C^a_i := E^a_i - \frac{1}{2}\epsilon^{abc}\epsilon_{ijk} e^j_b e^k_c \approx 0 \tag{37}$$

$$\mathcal{S} := \frac{\epsilon_i{}^{jk} E^a_j E^b_k \mathscr{D}_a K^i_b}{\sqrt{\det E}} \approx 0 \quad \text{(scalar constraint)}, \tag{38}$$

$$\mathcal{V}_a := E^a_i \mathscr{F}^i_{ab} \approx 0 \quad \text{(vector constraint)}, \tag{39}$$

$$\mathcal{G}_i := \mathscr{D}_a E^a_i \approx 0 \quad (SO(3) \text{ Gauss law}), \tag{40}$$

$$B_i := \epsilon_{ijk} E^{aj} K^k_a \approx 0 \quad , \tag{41}$$

$$\Pi^a_{i0} \approx 0. \tag{42}$$

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[2] Here we need to choose $e^0_\mu = n_\mu = (-N, 0, 0, 0)$ instead of letting $e_{\mu 0} = n_\mu$ because of the convention $\det(e_{\mu I}) > 0$, which is chosen to let e^0_μ to be future pointing.

2. Secondary constraints under time gauge

The consistency condition $\dot{M}_i^a \approx 0$ implies

$$\lambda_b^j \epsilon^{abc} \epsilon_{ijk} e_c^k + \cdots \approx 0 \tag{43}$$

which fixes the nine Lagrange multipliers λ_a^i. The consistency condition $\dot{C}_i^a \approx 0$ gives

$$-\rho_b^j \epsilon^{abc} \epsilon_{ijk} e_c^k + \cdots \approx 0, \tag{44}$$

which fixes the Lagrange multipliers ρ_a^i. The consistency conditions $\dot{\Pi}_{i0}^a \approx 0$ are best understood if we split the nine components of Π_{i0}^a as follows [8]

$$\Pi^i := \epsilon^{ijk} e_{aj} \Pi_{0k}^a, \quad \Pi_{ij} := e_{a(i} \Pi_{0j)}^a = \Pi_{ji}, \tag{45}$$

Now $\dot{\Pi}_{ij} \approx 0$ implies six secondary constraints

$$S_{ij} := \epsilon^{abc} e_{a(i} \mathscr{D}_b e_{cj)} \approx 0, \tag{46}$$

while $\dot{\Pi}_i$ implies

$$N^i - {}^3\!e^{-1} E^{ai} \partial_a N' + \cdots \approx 0, \tag{47}$$

which fixes the three Lagrange multipliers N^i. At this stage an important remark is in order. Notice that the six constraints $S_{ij} = 0$ together with the three Gauss law three $\mathscr{D}_a E_i^a \approx 0$ (40) are equivalent to the nine $\epsilon^{abc} \mathscr{D}_b e_c^i \approx 0$ which in turn can be more conveniently written as

$$\mathbb{D}_a^i := A_a^i - \Gamma_a^i(E) \approx 0, \tag{48}$$

where Γ_a^i is the spin connection compatible with the triad e_a^i. Therefore, the secondary constraints (46) and the (40) can be replaced by (48).

3. Reducibility of the constraints

Same as in the direct analysis, §II B 2, we can prove that the scalar constraint and the vector constraints are implied by other constraints and hence redundant.

Due to (48) $\mathscr{D}_{[a} e_{b]}^i \approx 0$ the scalar constraint can be re-written as

$$\mathcal{S} = \frac{\epsilon_i{}^{jk} E_j^a E_k^b \mathscr{D}_a K_b^i}{\sqrt{\det E}} \approx \mathscr{D}_a \left(\frac{\epsilon_i{}^{jk} E_j^a E_k^b K_b^i}{\sqrt{\det E}} \right) \approx 0 \tag{49}$$

where in the last equality we have used (41). The previous equation tell us that the scalar constraint is in fact implied by the constraints (48) and (41), or equivalently by (41), (46) and (40). A similar thing happens for the vector constraint. We first observe that $\mathscr{D}_{[a} e_{b]}^i \approx 0$ implies $\epsilon^{abc} \mathscr{D}_a \mathscr{D}_b e_c^i \approx 0$ from which we obtain (using the definition of the curvature strength) $\epsilon^{abc} \epsilon^i{}_{jk} \mathscr{F}_{ab}^j e_c^k \approx 0$. Using the constraint (37) one shows in a line that this implies

$$\mathscr{F}_{ab}^j E_i^a E_j^b = \mathcal{V}_a E_i^a \approx 0. \tag{50}$$

Using the (assumed) invertibility of E_i^a we conclude that the vector constraints $\mathcal{V}_b \approx 0$ are implied by the constraints (48) and (37). There are no more redundant constraints. Eliminating the redundant constraints the Hamiltonian can be written as

$$H_T = \int [N^i \epsilon_{ijk} E^{aj} K_a^k + M^i \mathscr{D}_a E_i^a + \alpha^{ij} S_{ij} + \lambda_i^a (e_a^i - \frac{\epsilon_{abc} E_j^b E_k^b \epsilon^{ijk}}{2\sqrt{\det(E)}}) + \rho_a^i M_i^a + \gamma^i \Pi_i + \gamma^{ij} \Pi_{ij}], \tag{51}$$

where we have added the secondary constraint S_{ij} with its Lagrange multiplier α^{ij} to the total Hamiltonian, and Π_i and Π_{ij} were defined in (45). Equivalently we can write

$$H_T = \int [N^i \epsilon_{ijk} E^{aj} K_a^k + \alpha_i^a (A_a^i - \Gamma_a^i(E)) + \lambda_i^a (e_a^i - \frac{\epsilon_{abc} E_j^b E_k^b \epsilon^{ijk}}{2\sqrt{\det(E)}}) + \rho_a^i M_i^a + \gamma^i \Pi_i + \gamma^{ij} \Pi_{ij}], \tag{52}$$

where we have replaced the Gauss law and S_{ij} by the equivalent condition (48).

4. Classification of constraints and solution of second class constraints

There are no further secondary constraints, we can thus proceed to their classification. Recalling the notation $B_i := \epsilon_{ijk} E^{aj} K^k_a$, and $C^i_a = e^i_a - \epsilon_{abc} E^b_j E^b_k \epsilon^{ijk}/(2\sqrt{\det(E)})$ (notice that instead of C^a_i defined in (37) we are using its inverse for convenience). Their algebra is summarized in the following matrix

$$
\begin{array}{c}
\\ M^b_j \\ C^j_b \\ B_j \\ \Pi^j \\ \mathbb{D}^b_j \\ \Pi_{kl}
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
M^a_i & C^i_a & B_i & \Pi^i & \mathbb{D}^a_i & \Pi_{ij}
\end{array}
\\
\left(\begin{array}{cccc|cc}
0 & -\delta^i_j \delta^b_a \delta^3_{xy} & 0 & 0 & 0 & 0 \\
\delta^j_i \delta^a_b \delta^3_{xy} & 0 & 0 & 0 & \{C^j_b, \mathbb{D}^a_i\} & 0 \\
0 & 0 & 0 & -2e\delta^i_j \delta^3_{xy} & \{B_j, \mathbb{D}^a_i\} & 0 \\
0 & 0 & 2e\delta^j_i \delta^3_{xy} & 0 & 0 & 0 \\
\hline
0 & \{\mathbb{D}^b_j, C^i_a\} & \{\mathbb{D}^b_j, B_i\} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}, \tag{53}
$$

where values are to be read in the weak sense, and we have used the known fact that (in the absence of boundaries) $\Gamma^i_a(E) = \delta F/\delta E^a_i$ for $F := \int E^a_i \Gamma^i_a$, which implies

$$
\{\mathbb{D}^a_i, \mathbb{D}^b_j\} = -\{\Gamma^a_i, A^b_j\} - \{A^a_i, \Gamma^b_j\} = -\frac{\delta \Gamma^a_i}{\delta E^j_b} + \frac{\delta \Gamma^b_j}{\delta E^i_a} = 0. \tag{54}
$$

The matrix Eq.(53) implies that the Π_{ij} are first class, the $\mathbb{D}^a_i$ can be made into first class by the addition of a suitable combination of the constraint in the upper left block consisting of M^a_i, C^i_a, B_i, and Π^i which are second class. Thus we have 15 first class constraints and 24 second class constraints. The phase space is spanned by (e^i_a, A^i_a, K^i_a) and their conjugate momenta so that it has 54 dimensions. Therefore, there are $54/2 - 15 - 24/2 = 0$ physical degrees of freedom. This result is consistent with the counting of the previous Subsections (II A) and (II B).

5. A partial reduction

In order to compare our model with the description of general relativity in terms of Ashtekar-Barbero variables it will be convenient to resolve the second class constraints above and gauge-fix the gauge symmetry generated by the first class constraints Π_{ij}. The first step is immediate as far as the constraints $C^i_a = 0$ and $M^a_i = 0$ are concerned. One just substitutes e^i_a using $C^i_a = 0$ everywhere and sets $M^a_i = 0$. By doing so the triad variables and their conjugate momenta are excluded from the phase space. Similarly for $B_i = 0$ and $\Pi^i = 0$ which removes three of the degrees of freedom in K^i_a (namely the B^i) and their conjugate momenta. In this way we are left with six remaining degrees of freedom in K^i_a. More precisely, these are given by $K^{ij} := E^{a(i}K^{j)}_a$. We can get rid of them by imposing the gauge fixing condition

$$
K^{ij} = 0 \tag{55}
$$

which fixes the gauge freedom generated by the six Π_{ij}. The reduced system is described by the action

$$
I_{red}[A, E] = \int dt \int_\Sigma \left[E^a_i \dot{A}^i_a - N^i \mathscr{D}_a E^a_i - \alpha^{ij} S_{ij} \right], \tag{56}
$$

or equivalently

$$
I_{red}[A, E] = \int dt \int_\Sigma \left[E^a_i \dot{A}^i_a - \alpha^a_i (A^i_a - \Gamma^i_a) \right]. \tag{57}
$$

The constraints are manifestly first class and the previous actions define a background independent $SU(2)$ connection gauge theory with no local degrees of freedom.

III. CONCLUSIONS

We have performed the canonical analysis of the theory (4) in three alternative ways. First the covariant phase space formulation of Subsection II A allows us to quickly learn that the theory is topological in the absence of boundaries. In Subsection II B we perform the Dirac analysis and obtain all the constraints and their classification. The counting of degrees of freedom is in agreement with the results of the covariant phase space formulation. However, second class constraints turn out to be rather complicated. The comparison with gravity in the Ashtekar-Barbero formulation suggested the analysis of the formulation of the field theory in the time gauge. With this partial gauge fixing, we find a surprisingly simple expression for the action of the model expressed in terms of a canonical pair (A^i_a, E^a_i) of an $SU(2)$ connection and its conjugate non Abelian electric field satisfying the usual Gauss (first class) constraints $\mathscr{D}_a E^a_i \approx 0$ plus six additional (first class) constraints stemming from 4-diffeo invariance of the original action plus two additional gauge symmetries that—from the perspective of the Holst action—kill the would-be-gravity degrees of freedom. These nine (first class) constraints can be concisely expressed by the conditions

$$A^i_a - \Gamma^i_a \approx 0$$

which are manifestly first class. From this fact, one could have had guessed at posteriori that action (57) is a consistent gauge theory with no local degrees of freedom. The extra merit of our analysis is to show that (57) comes indeed from (4).

We would like to stress a novel feature of the theory studied here. On the one hand it is a very simple model as it does not have any local degrees of freedom in the absence of boundaries. In this respect it shares a place with other topological theories in 4d such as BF theory. On the other hand, and this is a unique feature of this model, the field content of the theory is exactly the same as the one of general relativity in the first order formulation: namely the gravitational field e^I_a and the Lorentz connection ω^{IJ}_a. Moreover, the phase space of the theory can be described by $SU(2)$ connection variables just as in the gravity case. This may make this theory an interesting playground to test ideas relevant for gravity in 4d in a simpler context (in particular when it concerns quantization) .

Notice that all the quantization techniques of loop quantum gravity can be directly imported to this simple theory: the definition of the kinematical Hilbert space, the quantization of geometric operators such as area and volume, and the quantization techniques of Thiemann for the promotion of the constraints to quantum operators. For example one could promote the nine constraints above to operators by replacing Poisson brackets by commutators in the classical identity

$$A^i_a - \Gamma^i_a = -2\{\{H_E(1), V\}, A^i_a\},$$

were $H_E(1)$ is the so-called Euclidean Hamiltonian (see for instance eq. 10.3.7 and 10.3.16 in Thiemann's book [5]).

Our argument given in the introduction suggests that the theory studied here should play an important role in understanding the origin of black hole entropy. In the standard treatment of black hole entropy in LQG one quantizes gravity in a spacetime with a boundary at the location of the black hole event horizons (with appropriate boundary conditions defining a so-called isolated horizon [14]). Our analysis implies that in the limit $G \to \infty$ and $\gamma \to 0$ with $G\gamma$ held constant discussed in the introduction black hole entropy remains fixed, while the gravitational degrees of freedom in the bulk disappear. The results of section II A tell us that degrees of freedom might remain at the boundary. But it is precisely only boundary degrees of freedom that enter the standard calculation of black hole entropy. Therefore, all this strongly suggests that the origin of black hole entropy can, in this sense, be associated with excitations of our simple model.

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the appendix we discuss this point further and we exhibit a simple example of boundary condition leading to local degrees of freedom at the boundary.

Appendix A: Boundary degrees of freedom

In this appendix we explicitly exhibit examples of how the system described in this paper can have local degrees of freedom if the space-time considered contains a boundary where, by defining assumption of the variational principle, fields are allowed to vary while appropriate boundary conditions are satisfied. In the first example we simply start from equation (12) and require some extra conditions on the one forms e^I on the boundary. A possible way to define natural boundary conditions is to start from the symmetry content we want the theory to have at the boundary. We will assume that boundary manifold is foliated by a preferred family of two-surfaces H and that the space time foliation is arbitrary in the bulk but it is restricted to coincide with the preferred foliation of the boundary at the boundary, namely $H = \partial\Sigma$. We will work in the time gauge $e^0 = 0$ and require $SU(2)$ local transformations of the triad at the boundary—from now on denoted $G(SU(2))$—as well as $\mathrm{Diff}(H)$ to be gauge symmetries of the boundary fields. This implies that the pre-symplectic structure (12) will have to have null vectors associated to these transformations. The symmetry requirement will define for us boundary conditions for the given field content. Notice also that this is precisely the symmetry content of the isolated horizon boundary condition [6].

Let us start with $SU(2)$ transformations. Under an infinitesimal $SU(2)$ transformation parametrized by the field $\alpha \in su(2)$ the triad transforms as $\delta_\alpha e^i = [\alpha, e]^i$. This transformation is a gauge symmetry if for all α and arbitrary $\delta \in \Gamma_{\rm cov}$ the equation $\Omega(\delta_\alpha, \delta) = 0$, namely

$$-\,\Omega(\delta_\alpha, \delta) = \int_{H=\partial\Sigma} \delta_\alpha e^i \wedge \delta e_i = \int_H [\alpha, e]^i \wedge \delta e_i = \frac{1}{2}\int_H \delta(\epsilon_{ijk}\alpha^j e^k \wedge e^i) = 0. \tag{A1}$$

The previous equation tell us that, given the present field content, in order to preserve $SU(2)$ gauge invariance at the boundary we must impose the (zero area) boundary condition

$$\Sigma_i = \epsilon_{ijk} e^j \wedge e^k = 0. \tag{A2}$$

This boundary condition is certainly inappropriate for studies in the context of the black hole entropy, we will describe below a different alternative more suitable for such context. Notice that only two out of the tree constraints $\Sigma^i = 0$ are really independent. The next gauge symmetry we would like to impose is $\mathrm{Diff}(H)$. Under an infinitesimal diffeomorphisms parametrized by a vector field $v \in T(H)$ the triad transforms as $\delta_v e^i = d(v \lrcorner e^i) + v \lrcorner de^i$. Similarly to the previous case, the requirement $\Omega(\delta_v, \delta) = 0$ for all $\delta \in \Gamma_{\rm cov}$ becomes:

$$\begin{aligned}\Omega(\delta_v, \delta) &= \int_H \delta e^i \wedge \delta_v e_i = \int_H \delta e_i \wedge (d(v \lrcorner e^i) + v \lrcorner de^i) = \\ &= \int_H d(\delta e_i) \wedge (v \lrcorner e^i) - d(\delta e_i \wedge (v \lrcorner e^i)) + \delta e_i \wedge (v \lrcorner de^i) = \\ &= \int_H \delta(de_i (v \lrcorner e^i)) = 0,\end{aligned} \tag{A3}$$

where we have used the fact that $\delta e_i \wedge (v \lrcorner de^i) = (v \lrcorner \delta e_i) \wedge de^i$ in the last term of the second line, and have assumed $\partial H = 0$ in the last line. At first sight one would the conclude that $\mathrm{Diff}(H)$ are gauge symmetries of the system if and only if the following vector constraint is satisfied

$$V_a = e_{ai} de^i_{bc} \epsilon^{bc} = 0; \tag{A4}$$

however, if we recall the bulk equation of motion (5), the time gauge, and the gauge condition (55), we see that the previous constraint is implied by $\Sigma^i = 0$ as $V_a = e^i_a \Gamma^j_b e^k_c \epsilon_{ijk} \epsilon^{bc} = -\Gamma^j \Sigma_j = 0$. Therefore we conclude that the only constraints on boundary fields, necessary to preserve the symmetry content required in our example is given by (the two independent components of) the vanishing area constraint (A2). It is immediate to check that the vanishing area constraints are

indeed first class. The unconstrained phase space is parametrized by the 6 local fields e^i_a which implies a reduced phase space parametrized by two local fields, i.e. the system defined in this example has one local degree of freedom on the boundary. Notice that this is a kind of generalization of the Husain-Kuchar model [16], as those studied in [17].

We have seen that with the field content given above the symmetry requirement $G(SU(2)) \rtimes \mathrm{Diff}(H)$—the symmetry group of isolated horizons—implies that area vanishing constraint $\Sigma^i = 0$ and therefore this system cannot accommodate in any suitable way the black hole system that motivated the study of the theory considered in this work. However, this conclusion can be circumvented if one allows for additional field content at the horizon. In particular, if in addition one allows for an $SU(2)$ connection A^i to be an independent degree of freedom at the boundary then the considerations that lead to the result of [7] imply that, if the isolated horizon boundary condition $\Sigma^i = -(a/\pi)F^i(A)$ is satisfied, then $G(SU(2)) \rtimes \mathrm{Diff}(H)$ of the enlarged field system is gauge symmetry group of the system, and the presymplectic structure becomes

$$\Omega(\delta_1, \delta_2) = \int_H \frac{a}{2\pi} \delta_1 A_i \wedge \delta_2 A^i - \delta_1 e^i \wedge \delta_2 e_i, \tag{A5}$$

where the first term is a boundary term added in order to preserve gauge invariance in the presence of a non vanishing boundary area while the second term is the boundary term coming from the bulk. It is possible that the detail study of the quantization of this model could shed light on the nature of BH entropy in LQG. However, even when a lot is known about the quantization of the first term (given by an $SU(2)$ Chern-Simons theory of the kind studied in [18]) this is not an easy task as it would require the background independent quantization of the second term defining the dynamics of the e^i_a field about which, to our knowledge, little is known. We hope to be able to deepen the understanding of this model in the future.

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